MATH 105 101 Practice Problems

1. Find the Maclaurin series of the following functions:

   (a) $f(x) = \displaystyle\int_0^{x^2} \frac{1}{1+t}\,dt.$

   (b) $f(x) = \dfrac{1}{(1+2x)^2}.$

2. Let $f(x) = \int_x^2 e^{-t}\,dt$. Find an upper bound for the error of the approximation of $\displaystyle\int_3^5 f(x)\,dx$ using Simpson's Rule with $n = 6$. *Do not write down the approximation itself.*

3. Let $f(x) = \dfrac{3x+7}{x^2+3x-4}$.

   (a) Use the method of partial fraction to decompose $f(x)$ as a sum of simpler fractions.

   (b) Let $f(x) = \displaystyle\sum_{k=0}^{\infty} c_k x^k$ be the power series representation of $f(x)$ at $x = 0$. Find the expression for c_k.

4. Find and classify all the critical points of the function:

$$F(x,y) = \int_0^{xy+x} h(t)\,dt,$$

   where $h(t)$ is a differentiable function on $\mathbb{R}$ and $h(t) > 0$ for all t.

5. Determine if the following series converge:

   (a) $\displaystyle\sum_{k=1}^{\infty} (\arctan(k))^{1/k}.$

   (b) $\displaystyle\sum_{k=1}^{\infty} e^k e^{-e^k}.$

   (c) $\displaystyle\sum_{k=1}^{\infty} \frac{k\sqrt[4]{k^5+k}+2k^2}{3k^3-k+1}.$

   (d) $\displaystyle\sum_{k=2}^{\infty} \left(\frac{1}{\sqrt{k}} - \frac{1}{\sqrt{k-1}}\right).$

6. Let $F(x,y) = \displaystyle\int_x^{xy} h(t)\,dt$, where $h(t)$ is a continuous function on $\mathbb{R}$. Let $h(0) = 3$, find the gradient $\nabla F\,|_{(0,5)}$.

7. Use the method of Lagrange multipliers to find the maximum value of the function

$$f(x, y) = x + x^2 + 4y$$

on the circle $x^2 + x + y^2 + 2y = 1$.

8. Let $a_k = \dfrac{1}{k^2 + 7k + 12}$.

   (a) Use the method of partial fraction to decompose a_k as a sum of simpler fractions.

   (b) Find a formula for the n-th partial sum s_n of the series $\displaystyle\sum_{k=1}^{\infty} a_k$, where $s_n = \displaystyle\sum_{k=1}^{n} a_k$.

   (c) Evaluate the series $\displaystyle\sum_{k=1}^{\infty} a_k$.

9. Find the total area of the region bounded by the curve $y = x^2 - x - 6$ and the x-axis between $x = 0$ and $x = 5$. *Note that this is not the same as the net area of the region.*

10. Find the value of k such that $f(x) = k \arcsin(x)$ is a probability density function for a continuous random variable on $[0, 1]$.

11. Evaluate the following indefinite integrals:

   (a) $\displaystyle\int \frac{x}{\sqrt{2x+3}}\, dx$.

   (b) $\displaystyle\int \frac{e^x}{\sqrt{9 - e^{2x}}}\, dx$.

12. Evaluate the following definite integrals:

   (a) $\displaystyle\int_0^1 -\frac{1}{\sqrt{4 - x^2}}\, dx$.

   (b) $\displaystyle\int_1^6 -\sqrt{25 - (x-1)^2}\, dx$.

   (c) $\displaystyle\int_0^{\ln(2)} e^{-x} f'(e^{-x})\, dx$, given that $f(1) = 5$ and $f(0.5) = 6$.

13. Evaluate the following numerical series:

   (a) $\displaystyle\sum_{k=1}^{\infty} \frac{1}{\pi^{k+1} k!}$, given that $e^x = \displaystyle\sum_{k=0}^{\infty} \frac{x^k}{k!}$.

   (b) $\displaystyle\sum_{k=1}^{\infty} \frac{(-1)^{k+1} \pi^{2k+1}}{6^{2k+1}(2k+1)!}$, given that $\sin(x) = \displaystyle\sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{(2k+1)!}$.

(c) $\displaystyle\sum_{k=0}^{\infty} \frac{(-1)^k}{3^k(2k+1)}$, given that $\displaystyle\arctan(x) = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{2k+1}$ for $|x| < 1$.

14. Given that $\displaystyle e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!}$, find the 42-th derivative $f^{(42)}(0)$ where $f(x) = e^{(x^3)}$.

15. Solve the following initial value problem:

$$\frac{dy}{dx} = xe^{x^2 - \ln(y^2)}, \qquad y(0) = 0.$$

You may leave the solution in its implicit form.